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A NEW APPROACH TO THE TRANSFORMATION OF THE UNCONTROLLABLE AND UNOBSERVABLE PAIRS TO THEIR CANONICAL CONTROLLABLE AND OBSERVABLE FORMS OF LINEAR SYSTEMS

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Abstract – New approaches to the transformations of the uncontrollable and unobservable matrices of linear systems to their canonical forms are proposed. It is shown that the uncontrollable pair (*A*,*B*) and unobservable pair (*A*,*C*) of linear systems can be transform to their controllable (*A*,*B*) and observable (*A*,*C*) canonical forms by suitable choice of nonsingular matrix *M* satisfying the equation $M[A \quad B]=[\overline{A} \quad \overline{B}]$ It is also shown that by suitable choice of the gain matrix *K* of the feedbacks of the derivative of the state vector it is possible to reduced the descriptor system to the standard one.

Key words – controllability, observability, canonical form, descriptor, linear system, transformation.

1. INTRODUCTION

The concepts of the controllability and observability introduced by Kalman [8,9] have been the basic notions of the modern control theory. It well-known that if the linear system is controllable then by the use of state feedbacks it is possible to modify the dynamical properties of the closed-loop systems [1, 2, 5-14]. If the linear system is observable then it is possible to design an observer which reconstructs the state vector of the system [1, 2, 5-14]. Descriptor systems of integer and fractional order has been analyzed in [5-7,13,14]. The stabilization of positive descriptor fractional linear systems with two different fractional order by decentralized controller have been investigated in [13]. The eigenvalues assignment in uncontrollable linear continuous-time systems has been analyzed in [4].

In this paper new approaches to the transformations of the uncontrollable and unobservable linear systems will be proposed. In Section 2 some basic theorems concerning matrix equations with non-square matrices and their solutions are given. Transformations of the uncontrollable pairs to their canonical forms are presented in Section 3 and of the unobservable pairs in Section 4. Transformation of the controllable pairs in one canonical forms to other one is analyzed in Section 5. Concluding remarks are given in Section 6.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, I_n - the $n\times n$ identity matrix.

2. MATRIX EQUATIONS WITH NON-SQUARE MATRICES AND THEIR SOLUTIONS

Consider the matrix equation

$$PX = Q, \tag{2.1}$$

where $P\in\Re^{n\times m}$, $Q\in\Re^{n\times p}$ are given and $X\in\Re^{m\times p}$ is unknown matrix.

Theorem 2.1. The matrix equation (2.1) has a solution *X* if and only if

$$\mathrm{rank}[P \quad Q] = \mathrm{rank}P. \tag{2.2}$$

Proof follows immediately from the Kronecker-Cappelli Theorem [3].

Theorem 2.2. If the condition (2.2) is satisfied then the solution *X* of the equation (2.1) is given by

$$X = P_r Q, \tag{2.3}$$

where $P_r\in\Re^{m\times n}$ is the right inverse of the matrix *P* given by

$$P_r = P^T[AA^T]^{-1} + (I_n - P^T[PP^T]^{-1}P)K_1, \quad K_1\in\Re^{m\times n} \tag{2.4a}$$

or

$$P_r = K_2[PK_2]^{-1}, \quad K_2\in\Re^{m\times n} \tag{2.4b}$$

the matrix K_1 is arbitrary and K_2 is chosen so that $\det[AK_2]\neq 0$.

Proof. From (2.3) and (2.4a) we have

$$X = P^T[PP^T]^{-1}Q + (I_n - P^T[PP^T]^{-1}P)K_1Q. \tag{2.5}$$

Substituting (2.5) and (2.4a) we obtain

$$PX = PP^T[PP^T]^{-1}Q + (P - PP^T[PP^T]^{-1}P)K_1Q = Q. \tag{2.6}$$

Proof of (2.4b) is similar.

Consider the matrix equation

$$\overline{X}\overline{P} = \overline{Q}, \tag{2.7}$$

where $\overline{P} \in \Re^{m \times n}$, $\overline{Q} \in \Re^{p \times n}$ are given and $\overline{X} \in \Re^{p \times m}$ is unknown matrix.

Theorem 2.3. The matrix equation (2.7) has a solution *X* if and only if

$$\operatorname{rank}\begin{bmatrix} \overline{P} \\ \overline{Q} \end{bmatrix} = \operatorname{rank}\overline{P}\ . \tag{2.8}$$

Proof is similar (dual) to the proof of Theorem 2.2.

Theorem 2.4. If the condition (2.8) is satisfied then the solution of the equation (2.7) is given by

$$\overline{X} = \overline{Q}\overline{P}_l, \tag{2.9}$$

where the left inverse of the matrix $\overline{P}$ is given by

$$\overline{P}_l = [\overline{P}^T\overline{P}]^{-1}\overline{P}^T + K_1(I_m - \overline{P}[\overline{P}^T\overline{P}]^{-1}\overline{P}^T),\quad K_1 \in \Re^{n \times m} \text{ - arbitrary} \tag{2.10a}$$

or

$$\overline{P}_l = [K_2\overline{P}]^{-1}K_2,\quad K_2 \in \Re^{m \times m} \text{ -arbitrary} \tag{2.10b}$$

and the matrix K_2 is chosen so that $\det[K_2\overline{P}] \neq 0$.

Proof is similar (dual) to the proof of Theorem 2.2.

3. TRANSFORMATIONS OF THE UNCONTROLLABLE PAIRS TO THEIR CANONICAL FORMS

Consider the continuous-time linear system

$$\dot{x} = Ax + Bu, \tag{3.1a}$$

$$y = Cx, \tag{3.1b}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, $C \in \Re^{p \times n}$.

To simplify the notion we assume *m* = 1(single input systems).

Definition 3.1. The pair (*A*,*B*) is called in its canonical controllable form if

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{3.2a}$$

or

$$A_2 = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.2b}$$

Theorem 3.1. There exists a matrix *M* which transforms the uncontrollable pair (*A*,*B*) to its canonical forms (3.2) if and only if the pair (*A*,*B*) satisfies the condition

$$\operatorname{rank}[A \quad B] = n \tag{3.3}$$

Proof. From Theorem 2.1 it follows that there exists a nonsingular matrix M satisfying the equation

$$[A \quad B]M = [\overline{A}_k \quad \overline{B}_k] \text{ for } k = 1,2 \tag{3.4}$$

if and only if the condition (3.3) is satisfied. □

In this case the matrix M can be computed using the formula (2.10b) in the form

$$M = K\{[A \quad B]K\}^{-1} \tag{3.5}$$

where the matrix *K* is chosen such that $\det[A \quad B]K \neq 0$.

Note that the equation (3.4) for $m = 1$ has many solutions with singular and nonsingular matrix *M*.

Example 3.1.Consider the uncontrollable pair

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.6}$$

find the matrix $M \in \Re^{3\times 3}$ satisfying the equation

$$[A \quad B]M = [\overline{A} \quad \overline{B}] \tag{3.7}$$

for which the pair

$$\overline{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3.8}$$

is controllable.

The pair (3.6) satisfies the condition (3.3) since

$$\operatorname{rank}[A \quad B] = \operatorname{rank}[A \quad B]\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = 2 = n \tag{3.9}$$

Using (3.6), (3.8) and (3.5) for

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.10}$$

we obtain

$$M = K\{[A \quad B]K\}^{-1}[\overline{A} \quad \overline{B}]$$
$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0.5 & -1 \end{bmatrix}. \tag{3.11}$$

Note that the rank of the matrix (3.10) is 2.It is easy to show that the nonsingular matrix

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad M_{11} \in \Re^{n\times n}, \quad M_{22} \in \Re^{m\times m} \tag{3.12}$$

also satisfies the equation (3.4).

4. TRANSFORMATIONS OF THE UNOBSERVABLE PAIRS TO THEIR CANONICAL FORMS

To simplify the notation we assume $p = 1$ (single output systems).

Definition 2. [1,5,8,13,14] The pair (A,C) is called in its canonical observable form if

$$\hat{A}_1 = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}, \quad \hat{C}_1 = [0 \quad \dots \quad 0 \quad 1] \tag{4.1a}$$

or

$$\hat{A}_2 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, \quad \hat{C}_2 = [1 \quad 0 \quad \dots \quad 0]. \tag{4.1b}$$

Consider the continuous-time linear system (3.1) with $A \in \Re^{n\times n}$ and $C \in \Re^{p\times n}$ and the equation

$$[A^T \quad C^T]M^T = [\overline{A}^T \quad \overline{C}^T]. \tag{4.2}$$

Theorem 4.1. There exists a matrix $M \in \Re^{(n+p)\times(n+p)}$ which transforms the unobservable pair (A,C) to its canonical form (4.1) if and only if

$$\operatorname{rank}\begin{bmatrix} A \\ C \end{bmatrix} = n \tag{4.3}$$

Proof is similar (dual) to the proof of Theorem 3.1.

Example 4.1. Find the matrix $M \in \Re^{3\times 3}$ which transforms the unobservable pair

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad C = [1 \quad 0] \tag{4.4}$$

into observable one

$$\overline{A}=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix},\quad \overline{C}=[0\quad 1]\,. \tag{4.5}$$

Note that pair (4.4) satisfies the condition (4.3) since

$$\operatorname{rank}\begin{bmatrix}A\\ C\end{bmatrix}=\operatorname{rank}\begin{bmatrix}1 & 0\\ 2 & 1\\ 1 & 0\end{bmatrix}=2=n \tag{4.6}$$

Using (4.2), (4.4) and (4.5) we obtain the equation

$$\begin{bmatrix}1 & 2 & 1\\ 0 & 1 & 0\end{bmatrix}M^T=\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix} \tag{4.7}$$

and its solution has the form

$$M=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 1\\ -1 & 1 & -1\end{bmatrix}. \tag{4.8}$$

The matrix (4.8) is nonsingular.

Note that the presented approach with some small modifications can be also used to the pair $(\overline{A}^T,\overline{C}^T)$.

5. TRANSFORMATIONS OF THE PAIRS IN THE CANONICAL CONTROLLABLE AND OBSERVABLE FORMS

Consider the following two pairs $(\overline{A},\overline{B})$ and $(\hat{A},\hat{B})$ in canonical forms (3.2) and (4.1),respectively. We are looking for nonsingular matrix $M\in\Re^{(n+m)\times(n+m)}$ such that

$$[\overline{A}\quad \overline{B}]M=[\hat{A}\quad \hat{B}]. \tag{5.1}$$

Theorem 5.1. The pair $(\overline{A},\overline{B})$ in the canonical form (3.2) can be always transformed by (5.1) into the pair $(\hat{A},\hat{B})$ in the canonical form (4.1).

Proof. By Theorem 2.1 the equation (5.1) has a solution *M* if and only if

$$\operatorname{rank}\{[\overline{A}\quad \overline{B}],[\hat{A}\quad \hat{B}]\}=\operatorname{rank}[\overline{A}\quad \overline{B}] \tag{5.2}$$

and this condition is always satisfied if the pairs are in canonical forms.□

Example 5.1. Consider the controllable pairs in their canonical forms

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5.3}$$

and

$$\hat{A} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{5.4}$$

In this case the condition (5.2) takes the form

$$\operatorname{rank}\{[\bar{A} \quad \bar{B}],[\hat{A} \quad \hat{B}]\} = \operatorname{rank}\left\{\begin{bmatrix} 0 & 1 & 0 \\ -2 & -3 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -2 & 1 \\ 1 & -3 & 0 \end{bmatrix}\right\} = \operatorname{rank}\begin{bmatrix} 0 & 1 & 0 \\ -2 & -3 & 1 \end{bmatrix}. \tag{5.5}$$

It is easy to show that the pair (5.3) can be transformed into the pair (5.4) by the use of the matrix

$$M = \begin{bmatrix} -1 & 4 & -1 \\ 0 & -2 & 1 \\ -1 & -1 & 1 \end{bmatrix}. \tag{5.6}$$

Note that the matrix (5.6) is nonsingular and it is not unique.

6. CONCLUDING REMARKS

New approaches to the transformations of the uncontrollable pairs (A,B) and of the unobservable pairs (A,C) of linear systems by suitable choice of the transformation M satisfying the equations (3.3) have been proposed and illustrated by the simple numerical example. Transformations of the pairs (A,B) and (A,C) in their canonical forms have been also proposed and illustrated by the simple numerical example. The considerations can be extended to linear discrete-time systems and to the fractional orders linear systems. An open problem is an extension of these considerations to the different fraction orders linear systems.

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