Journal of Automation, Electronics and Electrical Engineering

Transformations Of The Discrete-Time Linear Systems To The Positive Asymptotically Stable Forms

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DOI: https://doi.org/10.24136/jeee.2024.005

Abstract – New approaches to the transformations of the discrete-time linear systems to their positive asymptotically stable canonical controllable (observable) forms is proposed. It is shown that if the matrix A of the system is nonsingular then the desired transformation matrix can be chosen in block diagonal form. Procedures for computation of the transformation matrices are proposed and illustrated by simple numerical examples.

Keywords – asymptotically stable, positive, continuous-time, linear, system, procedure

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine. An overview of state of art in positive systems theory is given in [3, 8, 9].

The concepts of the controllability and observability introduced by Kalman [12, 13] have been the basic notions of the modern control theory. It well-known that if the linear system is controllable then by the use of state feedbacks it is possible to modify the dynamical properties of the closed-loop systems [1, 3, 6, 7, 9-16, 19]. If the linear system is observable then it is possible to design an observer which reconstruct the state vector of the system [1-3, 6, 7, 9-16, 19]. Descriptor systems of integer and fractional order has been analyzed in [9, 17, 18]. The stabilization of positive descriptor fractional linear systems with two different fractional order by decentralized controller have been investigated in [18]. The eigenvalues assignment in uncontrollable linear continuous-time systems has been analyzed in [5]. New approaches to transformations of the linear continuous-time systems to the positive asymptotically stable have been proposed in [8].

In this paper new approaches to the transformations of the discrete-time linear systems to their positive asymptotically stable canonical controllable (observable) forms has been proposed. In Section 2 some basic definitions and theorems concerning linear standard continuous-time systems are recalled. A new approach to the transformations of the linear systems to their asymptotically stable controllable canonical forms is proposed in Section 3 and extended to observable canonical forms in Section 4.Concluding remarks are given in Section 5.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, I_n - the $n\times n$ identity matrix, *M* – the set of Metzler matrices (matrices with nonnegative off-diagonal entries).

2. PRELIMINARIES

Consider the discrete-time linear system

$$, x_{i+1} = Ax_i + Bu_i , \tag{2.1a}$$

$$y_i = Cx_i , \tag{2.1b}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

Theorem 2.1. [6] The solution of the equation (2.1a) has the form

$$x_i = A^i x_0 + \sum_{k=0}^{i-1} A^{i-k-1} Bu_k , \quad i = 0,1,... \tag{2.2}$$

Definition 2.1. [7] The system (2.1) is called (internally) positive if the state vector $x_i \in \Re_+^n$, output vector $y_i \in \Re_+^p$ for *i* = 0,1,... for all initial conditions $x_0 \in \Re_+^n$ and all inputs $u_i \in \Re_+^m$ for *i* = 0,1,...

Theorem 2.2. [7] The linear system (2.1) is positive if and only if

$$A \in \Re_+^{n\times n} , \; B \in \Re_+^{n\times m} , \; C \in \Re_+^{p\times n} . \tag{2.3}$$

Definition 2.2. [7-9] The positive system (2.1) is called reachable (controllable) in *n* steps if there exists an input sequence $u_i \in \Re_+^m$ for $i = 0,1,...,n-1$ which steers the state of the system from the zero initial condition to the final state $x_f = x_n$.

Theorem 2.3. [7-9] The linear system (2.1) is reachable if

$$\mathrm{rank}R = n \tag{2.4a}$$

and

$$R^T[RR^T] \in \Re_+^{nm\times n}. \tag{2.4b}$$

where

$$R = [B \quad AB \quad \dots \quad A^{n-1}B]. \tag{2.4c}$$

Remark 2.1. Single input (*m* = 1) positive system (2.1) is reachable in n steps if the matrix

$$R = [B \quad AB \quad \dots \quad A^{n-1}B] \in \Re_+^{n\times n} \tag{2.5}$$

is the permutation matrix.

Definition 2.3. The linear system (2.1) is observable in *n* steps if knowing its inputs $u_0, u_1, \dots, u_{n-1}$ and its outputs $y_0, y_1, \dots, y_{n-1}$ it is possible to find its unique initial state x_0.

It is well known [7, 12, 13] that the observability is the dual notion to the controllability.

Theorem 2.4. The single output (*p* = 1) system (2.1) is observable in n steps if the matrix

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \Re_+^{n\times 1} \tag{2.6}$$

is the permutation matrix.

Theorem 2.5. [4] (Kronecker-Cappelly). The matrix equation

$$PX = Q,\ P \in \Re^{n\times p},\ Q \in \Re^{n\times q} \tag{2.7}$$

has a solution X if and only if

$$\mathrm{rank}[P \quad Q] = \mathrm{rank}P \tag{2.8}$$

Theorem 2.6. [4] If the condition (2.8) is satisfied then the solution $X \in \Re^{p\times 1}$ of the matrix equation (2.7) for $P \in \Re^{n\times p}$ is given by

$$X = \{P^T[PP^T]^{-1} + (I_q - P^T[PP^T]^{-1}P)K_1\}Q \tag{2.9a}$$

or

$$X = K_2[PK_2]^{-1}Q \tag{2.9b}$$

where K_1 and K_2 are real matrices.

3. POSITIVE DISCRETE-TIME ASYMPTOTICALLY STABLE LINEAR SYSTEMS WITH CONTROLLABLE PAIRS (A,B)

For given pair (*A*,*B*) of the system (2.1) for *m* = 1 satisfying the condition

$$\mathrm{rank}[A \quad B] = n \tag{3.1}$$

find a nonsingular matrix

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \Re^{(n+1)\times(n+1)}, \quad M_{11} \in \Re^{n\times n}, \quad M_{22} \in \Re^{1\times 1} \tag{3.2}$$

such that

$$[A \quad B]M = [\overline{A} \quad \overline{B}] \tag{3.3}$$

where the pair (*A*,*B*) is positive and asymptotically stable.
It will be shown that if

$$\det A \neq 0 \tag{3.4}$$

then we may assume the matrix (3.2) in the form

$$M = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}. \tag{3.5}$$

In this case from (3.3) we have

$$AM_{11} = \overline{A} \tag{3.6}$$

and

$$M_{11} = A^{-1}\overline{A}\,. \tag{3.7}$$

From (3.3) and (3.5) we have

$$[A \quad B]\begin{bmatrix} M_{12} \\ M_{22} \end{bmatrix} = \overline{B}\,. \tag{3.8}$$

We choose M_{12} such that

$$\text{rank}[B \quad \overline{B} - AM_{12}] = \text{rank}B\,. \tag{3.9}$$

From Theorem 2.5 it follows that the equation

$$BM_{22} = \overline{B} - AM_{12} \tag{3.10}$$

has a solution M_{22} for given $\overline{B}$ and M_{12}. Therefore, the following theorem has been proved.

Theorem 3.1. If the condition (3.4) is satisfied then the matrix M can be chosen in the form (3.5), where M_{11} is given by (3.7) and M_{22} is a solution of the equation (3.10).

Remark 3.1. Note that if

$$\text{rank}[\overline{B} \quad B] = \text{rank}B \tag{3.11}$$

then we may assume $M_{12} = 0$ and the matrix M has block diagonal form. The matrix (3.5) can be computed by the use of the following procedure.

Procedure 1.

Step 1. Check the condition (3.4).

Step 2. Using (3.7) compute the matrix M_{11}.

Step 3. Choose the matrix M_{12} such that the condition (3.9) is satisfied.

Step 4. Using (3.10) compute the matrix M_{22} and the desired matrix (3.5).

Example 3.1. Consider the uncontrollable system (2.1a) with the matrices

$$A = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3.12}$$

Compute the matrix (3.5) such that

$$\bar{A} = \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.4 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{3.13}$$

Using Procedure 1 we obtain.

Step 1. The matrix *A* given by (3.12) is nonsingular.

Step 2. Using (3.7) and (3.12) we obtain

$$M_{11} = A_1^{-1}\bar{A} = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 0 & -0.15 \\ 0.2 & 0.55 \end{bmatrix}. \tag{3.14}$$

Step 3. In this case we choose

$$M_{12} = -\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \tag{3.15}$$

since

$$\bar{B} - AM_{12} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} M_{22}. \tag{3.16}$$

Step 4. The equation (3.10) for (3.15) has the form

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} M_{22} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.17}$$

and its solution

$$M_{22} = [1]. \tag{3.18}$$

The desired matrix M has the form

$$M = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix} = \begin{bmatrix} -0.2 & -0.1 & -0.5 \\ 0.3 & 0.7 & -0.5 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.19}$$

If det A = 0 and

$$\mathrm{rank}[A \quad B] = n \tag{3.20}$$

the from (3.3) we have

$$[A \quad B]\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} = \bar{A} \tag{3.21a}$$

and

$$[A \quad B]\begin{bmatrix} M_{12} \\ M_{22} \end{bmatrix} = \bar{B} \,. \tag{3.21b}$$

If the condition (3.20) is satisfied then by Theorem 2.5 the equations (3.21) have solutions which can be computed by the use of Theorem 2.6.

Therefore, we have the following Theorem:

Theorem 3.2. If $\det A \neq 0$ and the condition (3.20) is satisfied then there exists the nonsingular matrix M_{22} such that the pair (*A*,*B*) is the desired positive controllable form.
In this case the matrix *M* can be computed by the use of the following procedure.

Procedure 3.2.
Step 1. Check the conditions det $A = 0$ and (3.20).
Step 2. For the given matrices *A*, *B* and $\bar{A}$ using Theorem 2.5 find the solution of the equation (3.21a).
Step 3. For the given matrices *A*, *B* and $\bar{B}$ using Theorem 2.5 find the solution of the equation (3.21b).
Step 4. Find the desired matrix *M*.

Example 3.2. For the uncontrollable system (2.1a) with the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3.22}$$

compute the matrix *M* such that

$$\bar{A} = \begin{bmatrix} 0 & 0.1 \\ 0.2 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3.23}$$

Using Procedure 3.2 we obtain

Step 1. The matrix A is singular and the pair (3.22) satisfies the condition (3.20) since

$$\det A = \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} = 0, \quad \text{rank}[A \quad B] = \text{rank}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = 2 = n. \tag{3.24}$$

Step 2. In this case the equation (3.21a) has the form

$$[A \quad B]\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} = \overline{A} = \begin{bmatrix} 0 & 0.1 \\ 0.2 & 0 \end{bmatrix} \tag{3.25}$$

and using (2.9a) for

$$P = [A \quad B] = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad Q = \overline{A} = \begin{bmatrix} 0 & 0.1 \\ 0.2 & 0 \end{bmatrix} \tag{3.26a}$$

we obtain

$$\begin{aligned} \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} &= \{P^T[PP^T]^{-1} + (I_p - P^T[PP^T]^{-1}P)K\}Q \\ &= \left\{\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} + \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right)\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \\ k_{31} & k_{32} \end{bmatrix}\right\}\begin{bmatrix} 0 & 0.1 \\ 0.2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.2k_{12} & 0.1k_{11} \\ 0.2 & 0 \\ -0.2 & 0.1 \end{bmatrix} \quad for \quad k_{12} \neq 0. \end{aligned} \tag{3.26b}$$

Now let us consider the particular case of the equation

$$[A \quad B]\begin{bmatrix} M_{12} \\ M_{22} \end{bmatrix} = \overline{B} \tag{3.27}$$

when the following condition

$$\text{rank}[\overline{B} \quad B] = \text{rank}B \tag{3.28}$$

is satisfied. In this case we may assume $M_{12} = 0$ and from (3.27) we obtain

$$BM_{22} = \overline{B} \tag{3.29}$$

since by Theorem 2.5 the equation (3.27) has nonsingular solution M_{22}.

Therefore, when $\det A \neq 0$ and the condition (3.29) is satisfied then the matrix M satisfying (3.3) has the block diagonal form

$$M = \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix}. \tag{3.30}$$

Theorem 3.3. If $\det A \neq 0$ and the condition (3.28) is satisfied then the matrix M has the block diagonal form (3.30).

4. POSITIVE DISCRETE-TIME ASYMPTOTICALLY STABLE LINEAR SYSTEMS WITH OBSERVABLE PAIRS

In this Section the considerations of the previous Section 3 will be extended to the positive discrete-time asymptotically stable linear systems with the observable pairs (A,C).

For a given pair (A,C) of the system (2.1) satisfying the condition

$$\text{rank}\begin{bmatrix} A \\ C \end{bmatrix} = n \tag{4.1}$$

find a nonsingular matrix

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}, \quad N_{11} \in \Re^{n \times n}, \quad N_{22} \in \Re^{p \times p} \tag{4.2}$$

such that

$$N\begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} \overline{A} \\ \overline{C} \end{bmatrix}, \overline{A} \in \Re_+^{n \times n}, \overline{C} \in \Re_+^{p \times n} \tag{4.3}$$

where the pair $(\overline{A}, \overline{C})$ is observable, positive and asymptotically stable.

Applying the transposition to (4.3) we obtain

$$[A^T \quad C^T]N^T = [\overline{A}^T \quad \overline{C}^T] \tag{4.4}$$

where $A^T, C^T, \overline{A}^T, \overline{C}^T$ are the transposed matrices.

From comparison of (4.4) and (3.3) it follows that the Problem 1 for the pair (A,C) has been reduced to the problem 1 for the pair (A,C) of the system (2.1). Therefore, applying Theorem 3.1 to (4.4) we obtain the following theorem.

Theorem 4.1. If $\det A \neq 0$ then the matrix N^T can be chosen in the form

$$N^T = \begin{bmatrix} N_{11} & N_{12} \\ 0 & N_{22} \end{bmatrix} \tag{4.5}$$

where

$$N_{11} = (A^T)^{-1}\overline{A}^T \tag{4.6}$$

the matrix N_{12} is chosen such that

$$\mathrm{rank}[\overline{C}^T - A^T \quad N_{12}] = \mathrm{rank}C^T \tag{4.7}$$

and N_{22} is the solution of the equation

$$C^T N_{22} = \overline{C}^T - A^T N_{12} \tag{4.8}$$

If det$A$ = 0 then we have

$$[N_{11} \quad N_{21}]\begin{bmatrix} A \\ C \end{bmatrix} = \overline{A} \tag{4.9a}$$

and

$$[N_{21} \quad N_{22}]\begin{bmatrix} A \\ C \end{bmatrix} = \overline{C}\,. \tag{4.9b}$$

By Theorem 2.6 the equations (4.9) have solutions since the condition (4.1) is satisfied. In this case by Theorem 2.6 the equations (4.9) have solutions and the matrix N can be computed. Therefore, we have the following theorem:

Theorem 4.2. If det A = 0 and the condition (4.1) is satisfied then there exists a nonsingular matrix N such that the pair (A,C) is in the observable canonical form.

5. CONCLUDING REMARKS

New approaches to the transformations of the discrete-time linear systems to their positive asymptotically stable controllable and observable forms have been proposed. Necessary and sufficient conditions for the transformations have been established (Theorem 3.1-3.3, 4.1, 4.2). Procedures for computation of the transformation matrices have been given and illustrated by simple numerical examples. The approaches can be easily extended to the fractional orders continuous-time and discrete-time linear systems. An open problem is an extension of these considerations to the different fractional orders linear systems.

ACKNOWLEDGMENT

This work was carried out in the framework of work No. WZ/WE-IA/5/2023 and financed from the funds for science by the Polish Ministry of Science and Higher Education.

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