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SOME ANALYSIS PROBLEMS OF THE LINEAR SYSTEMS

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Abstract – Abstract: New approaches to the transformations of the uncontrollable and unobservable matrices of linear systems to their canonical forms are proposed. It is shown that the uncontrollable pair (*A*,*B*) and unobservable pair (*A*,*C*) of linear systems can be transform to their controllable $(\bar{A}, \bar{B})$ and observable $(\bar{A}, \bar{C})$ canonical forms by suitable choice of nonsingular matrix M satisfying the condition $M[A \quad B] = [\bar{A} \quad \bar{B}]$ and $[A \quad B]M = [\hat{A} \quad \hat{B}]$, respectively. It is also shown that by suitable choice of the gain matrix K of the feedbacks of the derivative of the state vector it is possible to reduce the descriptor system to the standard one.

Key words – controllability, observability, canonical form, descriptor, linear system.

INTRODUCTION

The concepts of the controllability and observability introduced by Kalman [8, 9] have been the basic notions of the modern control theory. It well-known that if the linear system is controllable then by the use of state feedbacks it is possible to modify the dynamical properties of the closed-loop systems [1, 2, 5-14]. If the linear system is observable then it is possible to design an observer which reconstruct the state vector of the system [1, 2, 5-14]. Descriptor systems of integer and fractional order has been analyzed in [6, 13]. The stabilization of positive descriptor fractional linear systems with two different fractional order by decentralized controller have been investigated in [13]. The eigenvalues assignment in uncontrollable linear continuous-time systems has been analyzed in [4].

In this paper new approaches to the transformations of the uncontrollable and unobservable linear systems will be proposed. In Section 2 some basic theorems concerning matrix equations with non-square matrices and their solutions are given. Transformations of the uncontrollable pairs to their canonical forms are presented in Section 3 and of the unobservable pairs in Section 4. Transformation of the controllable pairs in one canonical forms to other one is analyzed in Section 5. Elimination of the singularity in descriptor linear systems is considered in Section 6. Reduction of the descriptor linear systems to their standard forms by the use of the feedbacks is analyzed in Section 7. Concluding remarks are given in Section 8.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, I_n - the $n\times n$ identity matrix.

I. MATRIX EQUATIONS WITH NON-SQUARE MATRICES AND THEIR SOLUTIONS

Consider the matrix equation

$$PX = Q, \tag{1}$$

where $P\in\Re^{n\times m}$, $Q\in\Re^{n\times p}$ are given and $X\in\Re^{m\times p}$ is unknown matrix.

Theorem 1. The matrix equation (1) has a solution *X* if and only if

$$\mathrm{rank}[P \quad Q] = \mathrm{rank}P. \tag{2}$$

Proof follows immediately from the Kronecker-Cappelly Theorem [3].

Theorem 2. If the condition (2) is satisfied then the solution *X* of the equation (1) is given by

$$X = P_r Q, \tag{3}$$

where $P_r\in\Re^{m\times n}$ is the right inverse of the matrix *P* given by

$$P_r = P^T[PP^T]^{-1} + (I_n - P^T[PP^T]^{-1}P)K_1, \quad K_1\in\Re^{m\times n} \tag{4a}$$

or

$$P_r = K_2[PK_2]^{-1}, \quad K_2\in\Re^{m\times n} \tag{4b}$$

the matrix K_1 is arbitrary and K_2 is chosen so that $\det[AK_2]\neq 0$.

Proof. From (3) and (4a) we have

$$X = P^T[PP^T]^{-1}Q + (I_n - P^T[PP^T]^{-1}P)K_1Q \quad (5)$$

Substituting (5) into (3) we obtain

$$PX = PP^T[PP^T]^{-1}Q + (P - PP^T[PP^T]^{-1}P)K_1Q = Q \quad (6)$$

Proof of (4b) is similar. □

Consider the matrix equation

$$\overline{X}\overline{P} = \overline{Q}, \quad)$$

where $\overline{P} \in \Re^{m\times n}$, $\overline{Q} \in \Re^{p\times n}$ are given and $\overline{X} \in \Re^{p\times m}$ is unknown matrix.

Theorem 3. The matrix equation (7) has a solution *X* if and only if

$$\operatorname{rank}\begin{bmatrix} \overline{P} \\ \overline{Q} \end{bmatrix} = \operatorname{rank}\overline{P} \cdot \quad (8)$$

Proof is similar (dual) to the proof of Theorem 1.

Theorem 4. If the condition (8) is satisfied then the solution of the equation (7) is given by

$$\overline{X} = \overline{Q}\overline{P}_l, \quad (9)$$

where the left inverse of the matrix $\overline{P}$ is given by

$$\overline{P}_l = [\overline{P}^T\overline{P}]^{-1}\overline{P}^T + K_1(I_m - \overline{P}[\overline{P}^T\overline{P}]^{-1}\overline{P}^T),\quad K_1 \in \Re^{n\times m} \text{ - arbitrary} \quad (10a)$$

or

$$\overline{P}_l = [K_2\overline{P}]^{-1}K_2,\quad K_2 \in \Re^{m\times m} \text{ - arbitrary} \quad (10b)$$

and the matrix K_2 is chosen so that $\det[K_2\overline{P}] \neq 0$.

Proof is similar (dual) to the proof of Theorem 2.

II. TRANSFORMATIONS OF THE UNCONTROLLABLE PAIRS TO THEIR CANONICAL FORMS

Consider the continuous-time linear system

$$\dot{x} = Ax + Bu, \quad (11a)$$

$$y = Cx, \quad (11b)$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

To simplify the notion we assume *m* = 1 (single input systems).

Definition 1. The pair (*A*,*B*) is called in its canonical controllable form if

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix},\quad B_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (12a)$$

or

$$A_2 = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix},\quad B_2 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (12b)$$

Theorem 5. There exists a nonsingular matrix $M \in \Re^{n\times n}$ which transforms the uncontrollable pair (*A*,*B*) satisfying the condition

$$\operatorname{rank}[A \quad B] = n \quad (13)$$

to the canonical forms (12) if and only if

$$\operatorname{rank}\begin{bmatrix} A & B \\ A_k & B_k \end{bmatrix} = \operatorname{rank}[A \quad B] \text{ for } k = 1,2. \quad (14)$$

Proof. From Theorem1 it follows that there exists a nonsingular matrix $M \in \Re^{n\times n}$ satisfying the equation

$$M[A \quad B] = [\overline{A}_k \quad \overline{B}_k] \text{ for } k = 1,2 \quad (15)$$

if and only if the condition (14) is satisfied. □

If the condition (14) is satisfied then for the given matrices *A*,*B* and $\overline{A}$, $\overline{B}$ the matrix *M* can be computed by the use of the following procedure.

Procedure 1.

Step 1. Check the condition (14).The problem has a solution if and only if the condition (14) is satisfied.

Step 2. Using the equality $MB = \overline{B}$ find the corresponding column of the matrix *M*.

Step 3. Using the equality $MA = \overline{A}$ find the remaining columns of the matrix *M*.

The theorem will be illustrated by the following simple example.

Example 1. Find the matrix $M \in \Re^{2\times 2}$ satisfying (15) which transforms the pair

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (16a)$$

to their canonical form

$$\overline{A} = \begin{bmatrix} 0 & -a_0 \\ 1 & -a_1 \end{bmatrix},\quad \overline{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (16b)$$

Using Procedure 1 we obtain the following.

Step 1. The condition (14) is satisfied for the matrix $\overline{A}$ with $a_1 = 0$.

Step 2. From the equality

$$\overline{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = MB = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix} \quad (17)$$

we obtain: $m_{12} = 1$, $m_{22} = 0$.

Step 3. Taking into account (17) and

$$\overline{A}=MA=\begin{bmatrix}m_{11} & 1\\ m_{21} & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \quad (18)$$

we obtain: $m_{11}=m_{21}=1$.

Therefore, the desired nonsingular matrix M has the form

$$M=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}. \quad (19)$$

Remark 1. The approach based on the equation

$$M[A \quad B]=[\overline{A} \quad \overline{B}] \quad (20)$$

can be also used to transform the controllable pair (*A*,*B*) to the desired standard controllable form $(\overline{A},\overline{B})$.

The procedure will be shown on the following simple example.

Example 2. For the controllable pair

$$A=\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix},\quad B=\begin{bmatrix}0\\ 1\end{bmatrix} \quad (21)$$

find the matrix $M\in\Re^{2\times 2}$ satisfying the equality (20) such that the pair $(\overline{A},\overline{B})$ the desired canonical form

$$\overline{A}=MA=\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix},\quad \overline{B}=MB=\begin{bmatrix}1\\ 0\end{bmatrix}. \quad (22)$$

From the equality

$$\overline{B}=MB=\begin{bmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix} \quad (23)$$

we have $m_{12}=1$, $m_{22}=0$.

Using (22) we obtain

$$\overline{A}=MA=\begin{bmatrix}m_{11} & 1\\ m_{21} & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix} \quad (24)$$

and $m_{11}=-1,\quad m_{21}=1$.

Therefore, the desired matrix *M* has the form

$$M=\begin{bmatrix}-1 & 1\\ 1 & 0\end{bmatrix}. \quad (25)$$

III. TRANSFORMATIONS OF THE UNOBSERVABLE PAIRS TO THEIR CANONICAL FORMS

To simplify the notation we assume *p* = 1(single output systems).

Definition 2. The pair (*A*,*C*) is called in its canonical observable form if

$$\hat{A}_1=\begin{bmatrix}0 & 0 & \dots & 0 & -a_0\\ 1 & 0 & \dots & 0 & -a_1\\ 0 & 1 & \dots & 0 & -a_2\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 1 & -a_{n-1}\end{bmatrix},\quad \hat{C}_1=[0 \quad \dots \quad 0 \quad 1] \quad (26a)$$

or

$$\hat{A}_2=\begin{bmatrix}0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & 1\\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1}\end{bmatrix},\quad \hat{C}_2=[1 \quad 0 \quad \dots \quad 0] \quad (26b)$$

Theorem 6. There exists a nonsingular matrix $M\in\Re^{n\times n}$ which transforms the unobservable pair (*A*,*C*) satisfying the condition

$$\text{rank}\begin{bmatrix}A\\ C\end{bmatrix}=n \quad (27)$$

to the canonical forms (26) if and only if

$$\text{rank}\begin{bmatrix}A & \hat{A}_k\\ C & \hat{C}_k\end{bmatrix}=\text{rank}\begin{bmatrix}A\\ C\end{bmatrix} \text{ for k=1,2.} \quad (28)$$

Proof is similar (dual) to the proof of Theorem 5.

If the condition (28) is satisfied then for the given matrices *A*, *C* and $\hat{A},\hat{B}$ the matrix $\hat{M}$ can be computed by the use of the following procedure.

Procedure 2.

Step 1. Check the condition (28). The problem has a solution if and only if the condition (28) is satisfied.

Step 2. Using the equality $C\hat{M}=\hat{C}$ find the corresponding column of the matrix $\hat{M}$.

Step 3. Using the equality $A\hat{M}=\hat{A}$ find the remaining columns of the matrix $\hat{M}$.

The procedure will be illustrated by the following simple example.

Example 3. Find the matrix $\hat{M}\in\Re^{2\times 2}$ satisfying (28) which transforms the pair

$$A=\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix},\quad C=[1 \quad 0] \quad (29)$$

to their canonical form

$$\hat{A}=\begin{bmatrix}0 & 1\\ 1 & -3\end{bmatrix},\quad \hat{C}=[0 \quad 1]. \quad (30)$$

Using Procedure 2 we obtain the following.

Step 1. The condition (28) is satisfied since

$$\text{rank}\begin{bmatrix}A & \hat{A}\\ C & \hat{C}\end{bmatrix}=\text{rank}\begin{bmatrix}1 & 0 & 0 & 1\\ -1 & 1 & 1 & -3\\ 1 & 0 & 0 & 1\end{bmatrix}=\text{rank}\begin{bmatrix}1 & 0\\ -1 & 1\\ 1 & 0\end{bmatrix}=\text{rank}\begin{bmatrix}A\\ C\end{bmatrix}. \quad (31)$$

Step 2. From the equality

$$C\hat{M}=[1 \quad 0]\begin{bmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix}=[0 \quad 1] \quad (32)$$

we obtain: $m_{11}=0$, $m_{12}=1$.

Step 3. Taking into account (29) and

$$A\hat{M} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \quad (33)$$

we obtain: $m_{21} = 1$, $m_{22} = -2$.

Therefore, the desired nonsingular matrix $\hat{M}$ has the form

$$\hat{M} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}. \quad (34)$$

IV. TRANSFORMATIONS OF THE CONTROLLABLE PAIRS TO THEIR CANONICAL FORMS

Consider the following two pairs $(\overline{A}, \overline{B})$ and $(\hat{A}, \hat{B})$ in canonical forms (26). We are looking for nonsingular matrix $M \in \Re^{n\times n}$ such that

$$M[\overline{A} \quad \overline{B}] = [\hat{A} \quad \hat{B}]. \quad (35)$$

Theorem 7. The pair $(\overline{A}, \overline{B})$ can be transformed by (35) into pair $(\hat{A}, \hat{B})$ if and only if

$$\text{rank}\begin{bmatrix} \overline{A} & \overline{B} \\ \hat{A} & \hat{B} \end{bmatrix} = \text{rank}[\overline{A} \quad \overline{B}]. \quad (36)$$

Proof. By Theorem 1 the equation (35) has a solution *M* if and only if the condition (36) is satisfied. □

Now we apply Theorem 7 to the pair $(\overline{A}, \overline{B})$ in their canonical form (12) and we obtain the following theorem.

Theorem 8. The pair (12a) cannot be transformed into pair (12b) by the nonsingular matrix $M \in \Re^{n\times n}$ satisfying (35).

Proof. Applying the condition (36) to the pairs (12) we obtain

$$\text{rank}\begin{bmatrix} \overline{A} & \overline{B} \\ \hat{A} & \hat{B} \end{bmatrix} = \text{rank}\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} & 1 \\ 0 & 0 & 0 & \dots & 0 & -a_0 & 1 \\ 1 & 0 & 0 & \dots & 0 & -a_1 & 0 \\ 0 & 1 & 0 & \dots & 0 & -a_2 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -a_{n-1} & 0 \end{bmatrix} = n+1 \quad (37)$$

and

$$\text{rank}[\overline{A} \quad \overline{B}] = \text{rank}\begin{bmatrix} 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 \\ -a_0 & -a_1 & \dots & -a_{n-2} & -a_{n-1} & 1 \end{bmatrix} = n \quad . \quad (38)$$

Therefore, the pair (12a)cannot be transformed to the pair(12b) by (35).

Example 3. Consider the controllable pairs in their canonical forms

$$\overline{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (39)$$

and

$$\hat{A} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (40)$$

In this case the condition (36) takes the form

$$\text{rank}\begin{bmatrix} \overline{A} & \overline{B} \\ \hat{A} & \hat{B} \end{bmatrix} = \text{rank}\begin{bmatrix} 0 & 1 & 0 \\ -2 & -3 & 1 \\ 0 & -2 & 1 \\ 1 & -3 & 0 \end{bmatrix} = 3 \quad (41)$$

and

$$\text{rank}[\overline{A} \quad \overline{B}] = \text{rank}\begin{bmatrix} 0 & 1 & 0 \\ -2 & -3 & 1 \end{bmatrix} = 2. \quad (42)$$

Therefore, does not exists the nonsingular matrix $M \in \Re^{2\times 2}$ which transforms the pair (39) into the pair (40).

Similar results we obtain for the observable pairs (*A*,*C*).

Theorem 9. The pairs pair $(\overline{A}, \overline{C})$ can be transformed by the matrix $N \in \Re^{n\times n}$

$$\begin{bmatrix} \overline{A} \\ \overline{C} \end{bmatrix} N = \begin{bmatrix} \hat{A} \\ \hat{C} \end{bmatrix} \quad (43)$$

to observable the pair $(\overline{A}, \overline{C})$ if and only if

$$\text{rank}\begin{bmatrix} \overline{A} & \hat{A} \\ \overline{C} & \hat{C} \end{bmatrix} = \text{rank}\begin{bmatrix} \overline{A} \\ \overline{C} \end{bmatrix}. \quad (44)$$

Proof is similar(dual) to the proof of Theorem 8.

V. ELIMINATION OF THE SINGULARITY IN DESCRIPTOR LINEAR SYSTEMS

Consider the continuous-time linear system

$$E\dot{x} = Ax + Bu, \quad (45)$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$ are the state, input and output vectors and $E, A \in \Re^{n\times n}$ $B \in \Re^{n\times m}$.

It is assumed that

$$\det E = 0 \text{ and } \det[Es - A] \neq 0, s \in C$$

(the field of complex numbers). (46)

We are looking for the matrix $M \in \Re^{n\times n}$ satisfying the equality

$$M[E \quad A \quad B] = [I_n \quad \overline{A} \quad \overline{B}] \quad (47)$$

which eliminate the singularity of the system (45).

Note that by Theorem 1 there exists the nonsingular matrix M satisfying (47) if and only if the condition

$$\text{rank}\begin{bmatrix} E & A & B \\ I_n & \overline{A} & \overline{B} \end{bmatrix} = \text{rank}[E \quad A \quad B] \quad (48)$$

is satisfied.

From (47) we have

$$ME = I_n \cdot \quad (49)$$

Note that for nonsingular matrix *M* the equation (49) has no singular solution *E*.

Therefore we have the following conclusion. By suitable choice of the matrix *M* it is not possible to transform the descriptor system (45) to the standard one of the form

$$\dot{x} = \overline{A}x + \overline{B}u \,, \quad (50)$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$ are the state, input and output vectors and $\overline{A} \in \Re^{n \times n}$, $\overline{B} \in \Re^{n \times m}$.

VI. Reduction of the Descriptor Linear Systems to Standard Ones by Feedbacks

Consider the descriptor system(45) with feedbacks of the derivative of the state vector shown in Fig.1.

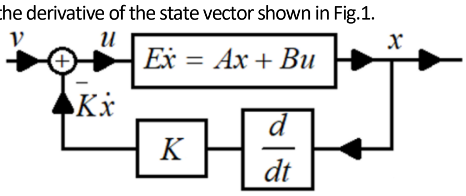

Fig.1. Descriptor system with feedback

Substituting the equality

$$u = v - K\dot{x} \text{ (}v\text{- the new input)} \quad (51)$$

into the equation

$$E\dot{x} = Ax + Bu \quad (52)$$

we obtain

$$E\dot{x} = Ax + B(v - K\dot{x}) \quad (53)$$

and

$$(E + BK)\dot{x} = Ax + Bv \cdot \quad (54)$$

The feedback matrix $K \in \Re^{m \times n}$ is chosen so that the matrix

$$F = E + BK = [E \quad B]\begin{bmatrix} I_n \\ K \end{bmatrix} \quad (55)$$

is nonsingular.

Note that there exists the matrix *K* such that the matrix *F* is nonsingular if and only if

$$\text{rank}[E \quad B] = n \cdot \quad (56)$$

Note that the equation (55) by Theorem 1 has the solution $\begin{bmatrix} I_n \\ K \end{bmatrix}$ if and only if

$$\text{rank}[E \quad B \quad F] = \text{rank}[E \quad B] \quad (57)$$

and this condition is satisfied if and only if (56) holds.

Therefore, the following theorem has been proved.

Theorem 10. There exists the matrix *K* such that the matrix *F* is nonsingular if and only if the condition (56) is satisfied.

For nonsingular matrix *F* from(54) we have

$$\dot{x} = \overline{A}x + \overline{B}v \,, \quad (58)$$

where

$$\overline{A} = F^{-1}A, \quad \overline{B} = F^{-1}B \cdot \quad (59)$$

Example 4. Consider the system (52) with the matrices

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (60)$$

which satisfies the condition (56) since

$$\text{rank}[E \quad B] = \text{rank}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 2 \cdot \quad (61)$$

By Theorem 10 there exists the feedback matrix $K = [k_1 \quad k_2]$ such that the matrix

$$F = E + BK = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}[k_1 \quad k_2] = \begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix} \quad (62)$$

is nonsingular. In this case the matrix (62) is nonsingular if $k_2 = 0$ and k_1 is nonzero. For $k_1 = 1, k_2 = 0$ we have

$$K = [1 \quad 0] \text{ and } F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (63)$$

and

$$\overline{A} = F^{-1}A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}, \quad (64)$$

$$\overline{B} = F^{-1}B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

VII. Conclusions

Two approaches to the transformations of the uncontrollable and unobservable linear systems to their canonical forms has been proposed (Theorems 5 and 6) and procedures for calculation of transformation matrices have been given (Procedures 1 and 2).The procedures have been illustrated by simple numerical examples. It has been shown that the pair (12a) Carnot been transformed to the pair (12b) by the nonsingular matrix *M* satisfying (35) (Theorem 7). Necessary and sufficient conditions have been established for the reduction of the descriptor linear systems to their standard forms(Theorem 8). The considerations can be extended to the discrete-time linear systems and to the fractional orders linear systems. An open problem is an extension of these approaches to the different orders linear systems.

WYBRANE ZAGADNIENIA ANALIZY UKŁADÓW LINIOWYCH

Zaproponowano nowe podejścia do transformacji niesterowalnych i nieobserwowalnych macierzy układów liniowych do ich postaci kanonicznych. Wykazano, że niesterowalna para (*A*,*B*) i nieobserwowalna para (*A*,*C*) układów liniowych może być przekształcona do ich postaci kanonicznych sterowalnych i obserwowalnych prze odpowiedni dobór nieosobliwej macierzy M spełniającej warunki
$M[A \quad B] = [\bar{A} \quad \bar{B}]$ i $[A \quad B]M = [\hat{A} \quad \hat{B}]$.
Pokazano, że przez odpowiedni dobór macierzy *K* sprzężenia zwrotnego od pochodnej wektora stanu jest możliwa redukcja układu deskryptorowego do układu standardowego.

Słowa kluczowe: sterowalność, obserwowalność, postać kanoniczna, układ deskryptorowy, układ liniowy

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