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A New Approach To The Stabilization Problem Of Linear Systems

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Abstract – A new approach for the stabilization of the linear continuous-time and discrete-time systems is proposed. The desired asymptotically stable state matrices of the linear systems are obtained by pre-multiplication and post-multiplication of the system matrix by suitable square matrix. Procedures for computation of the matrices are given and illustrated by simple numerical examples.

Keywords – approach, stabilization, continuous-time ,discrete-time, linear, system, procedure

1. Introduction

The concepts of the controllability and observability introduced by Kalman [9,10] have been the basic notions of the modern control theory. It well-known that if the linear system is controllable then by the use of state feedbacks it is possible to modify the dynamical properties of the closed-loop systems [1, 2, 4-15]. If the linear system is observable then it is possible to design an observer which reconstruct the state vector of the system [1, 2, 5-15]. Descriptor systems of integer and fractional order has been analyzed in [6,14]. The stabilization of positive descriptor fractional linear systems with two different fractional order by decentralized controller have been investigated in [14]. In [4] it has been shown that it is possible to assign arbitrarily the eigenvalues of the closed-loop system with state feedbacks if $\mathrm{rank}[A \quad B] = n$.

In this paper a new approach to the stabilization of the linear continuous-time and discrete-time systems will be proposed.

In section 2 some basic definitions and theorems concerning controllability, observability and stability of linear systems are recalled. In Section 3 the Frobenius canonical forms of the matrices are given and applied in algebraic matrix equations. New methods for stabilization of the linear systems by multiplication of the system matrix by suitable matrices is proposed in Section 4 for continuous-time and in Section 5 for discrete-time systems. Concluding remarks will be given in Section 6.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, I_n - the $n\times n$ identity matrix, the upper index T denotes the transposition of the matrix.

2. PRELIMINARIES

Consider The linear continuous-time system

$$\dot{x} = Ax + Bu, \tag{2.1a}$$

$$y = Cx, \tag{2.1b}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

Definition 2.1. The system (2.1) (the pair (*A*,*B*)) is called controllable in the interval $t \in [0, t_f]$ if there exists an input *u*(*t*), $t \in [0, t_f]$ which steers the system from the initial state $x_0 = x(0)$ to the given final state $x_f = x(f)$.

Theorem 2.1. The system (2.1) (the pair (*A*,*B*)) is controllable if and only if one of the following equivalent conditions is satisfied:

1) Kalman condition

$$\operatorname{rank}[B \quad AB \quad \dots \quad A^{n-1}B] = n \tag{2.2}$$

2) Hautus conditio

$$\operatorname{rank}[I_n s - A \quad B] = n \text{ for } s \in \mathrm{C} \text{ (the field of complex numbers)} \tag{2.3}$$

Definition 2.2. The autonomous linear system

$$\dot{x} = Ax, \;\; A \in \Re^{n\times n} \tag{2.4}$$

is asymptotically stable if

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} e^{At} x_0 = 0 \tag{2.5}$$

for all finite $x_0 \in \Re^n$.

Theorem 2.2. The system (2.4) is asymptotically stable if and only if

$$\operatorname{Re} s_i < 0 \text{ for } i = 1,\dots,n \tag{2.6}$$

where s_i are the eigenvalues of the matrix *A* and

$$\det[I_n s - A] = (s - s_1)(s - s_2)\dots(s - s_n) \tag{2.7}$$

Consider the linear discrete-time system

$$x_{i+1} = Ax_i + Bu_i\,, \tag{2.8a}$$

$$y_i = Cx_i\,, \tag{2.8b}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

Definition 2.3. The system (2.8) (the pair (*A*,*B*)) is called controllable in the interval [0,*q*] if there exists an input u_i for $i = 0,1,\dots,q-1$ which steers the system from the initial state x_0 to the given final state x_q.

Theorem 2.3. The system (2.8) (the pair (*A*,*B*)) is controllable if and only if one of the following equivalent conditions is satisfied:

Kalman condition

$$\mathrm{rank}[B \quad AB \quad \dots \quad A^{n-1}B] = n \tag{2.9}$$

Hautus condition

$$\mathrm{rank}[I_n z_i - A \quad B] = n \text{ for } z_i \in \mathrm{C} \text{ (the field of complex numbers)} \tag{2.10}$$

Definition 2.4. The autonomous system

$$x_{i+1} = Ax_i,\ \ A \in \Re^{n\times n} \tag{2.11}$$

is asymptotically stable if

$$\lim_{i\to n} x_i = \lim_{i\to\infty} A^i x_0 = 0 \tag{2.12}$$

for all finite $x_0 \in \Re^n$.

Theorem 2.4.The system (2.8) is asymptotically stable if and only if

$$|z_i| < 1 \text{ for } i = 1,\dots,n \tag{2.13}$$

where z_i are the eigenvalues of the matrix *A*.

From comparison of (2.6) and (2.11) we have the following remark.

Remark 2.1. The asymptotic stability of the continuous –time systems (2.1) depends on the phase of the eigenvalues s_i and of the discrete-time systems (2.8) on the absolute values (modulus) of the eigenvalues z_i.

3. ALGEBRAIC EQUATIONS WITH MATRICES IN FROBENIUS CANONICAL FORMS

Definition 3.1.The matrix $A \in \Re^{n\times n}$ has the Frobenius canonical form if it has one of the following forms [4]

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, \quad A_2 = A_1^T = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \dots & -a_1 & -a_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \quad A_4 = A_3^T = \begin{bmatrix} -a_{n-1} & 1 & 0 & \dots & 0 \\ -a_{n-2} & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -a_1 & 0 & 0 & 0 & 1 \\ -a_0 & 0 & 0 & \dots & 0 \end{bmatrix}. \tag{3.1}$$

Lemma 3.1. The inverse matrices of the matrices (3.1) have also the Frobenius canonical forms

$$A_1^{-1} = \begin{bmatrix} -\frac{a_1}{a_0} & -\frac{a_2}{a_0} & \dots & -\frac{a_{n-1}}{a_0} & -\frac{1}{a_0} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \quad A_2^{-1} = \begin{bmatrix} -\frac{a_1}{a_0} & 1 & 0 & \dots & 0 \\ -\frac{a_2}{a_0} & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{a_{n-1}}{a_0} & 0 & 0 & 0 & 1 \\ -\frac{1}{a_0} & 0 & 0 & \dots & 0 \end{bmatrix},$$

$$A_3^{-1} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 1 \\ -\frac{1}{a_0} & -\frac{a_{n-1}}{a_0} & -\frac{a_2}{a_0} & \dots & -\frac{a_1}{a_0} \end{bmatrix}, \quad A_4^{-1} = \begin{bmatrix} 0 & 0 & \dots & 0 & -\frac{1}{a_0} \\ 1 & 0 & \dots & 0 & -\frac{a_{n-1}}{a_0} \\ 0 & 1 & \dots & 0 & -\frac{a_{n-2}}{a_0} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & -\frac{a_1}{a_0} \end{bmatrix}. \tag{3.2}$$

Lemma 3.2. If the matrices *A* and *B* in the equation

$$AX_1 = B,\ \ A, B \in \Re^{n\times n},\ \det A \neq 0 \tag{3.3}$$

have the same Frobenius canonical forms then its solution $X_1 \in \Re^{n\times n}$ has the following form

$$X_1 = A^{-1}B = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \dots & \bar{x}_{n-1} & \bar{x}_n \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} \tag{3.4a}$$

where

$$\bar{x}_1 = \frac{b_0}{a_0},\ \ \bar{x}_2 = \frac{b_1 - a_1}{a_0}, \dots, \bar{x}_{n-1} = \frac{b_{n-2} - a_{n-2}}{a_0},\ \ \bar{x}_n = \frac{b_{n-1} - a_{n-1}}{a_0}. \tag{3.4b}$$

Proof. Let the matrices *A* and *B* have the same Frobenius canonical form

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix},$$
$$B = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 1 \\ -b_0 & -b_1 & -b_2 & \dots & -b_{n-1} \end{bmatrix},\quad \det A = a_0 \neq 0 \tag{3.5}$$

then from (3.3) using (3.2) we obtain

$$X = A^{-1}B = \begin{bmatrix} -\frac{a_1}{a_0} & -\frac{a_2}{a_0} & \dots & -\frac{a_{n-1}}{a_0} & -\frac{1}{a_0} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 1 \\ -b_0 & -b_1 & -b_2 & \dots & -b_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \dots & \bar{x}_{n-1} & \bar{x}_n \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}, \tag{3.6}$$

where $\bar{x}_i$, $i = 1,\dots,n$ is given by (3.4b).

Lemma 3.3.If the matrices *A* and *B* in the equation

$$X_2 A = B, \quad A, B \in \Re^{n\times n}, \ \det A \neq 0 \tag{3.7}$$

have the same Frobenius canonical forms then its solution has the form

$$X_2 = BA^{-1} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \\ \hat{x}_1 & \hat{x}_2 & \dots & \hat{x}_{n-1} & \hat{x}_n \end{bmatrix}. \tag{3.8}$$

Proof is similar (dual) to the proof of Lemma 3.2.

Example 3.1. Find the solution X to the matrix equations

$$AX_1 = B, \tag{3.9}$$

$$X_2 A = B, \tag{3.10}$$

for the matrices

$$A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 2 & 3 & 0\end{bmatrix},\quad B=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -12 & -16 & -7\end{bmatrix}. \tag{3.11}$$

Using (3.9) and (3.11) we obtain

$$X_1=A^{-1}B=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 2 & 3 & 0\end{bmatrix}^{-1}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -12 & -16 & -7\end{bmatrix}=\begin{bmatrix}-6 & -9.5 & -3.5\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{3.12}$$

and for (3.10) and (3.11)

$$X_2=BA^{-1}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -12 & -16 & -7\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 2 & 3 & 0\end{bmatrix}^{-1}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 2 & -7 & -6\end{bmatrix} \tag{3.13}$$

Note that the solutions X_1 and X_2 have different forms.

Remark 3.1. Lemmas 3.2 and 3.3 can be extended to the remaining Frobenius canonical forms of the matrices *A* and *B*.

4. STABILIZATION OF THE CONTINUOUS-TIME LINEAR SYSTEMS

Let S_M be the set of real matrices $A\in\Re^{n\times n}$ which by the similarity transformation

$$PAP^{-1}\in S_M,\quad \det P\neq 0 \tag{4.1}$$

can be reduced to the Frobenius canonical forms (3.1).

For given unstable matrix $A\in\Re^{n\times n}$ we are looking for the nonsingular matrix $M\in\Re^{n\times n}$ such that

$$MA=\overline{A} \tag{4.2}$$

or

$$AM=\overline{A} \tag{4.3}$$

where $\overline{A}\in\Re^{n\times n}$ is asymptotically stable (Hurwitz matrix) in the Frobenius canonical form (3.1).

From (4.2) and (4.3) we have

$$M = \overline{A}A^{-1} \tag{4.4}$$

and

$$M = A^{-1}\overline{A} \tag{4.5}$$

By Lemma 3.2 and 3.3 the matrix *M* has the special forms (3.4) and (3.8), respectively. Therefore, the following theorem has been obtained.

Theorem 4.1.If the given matrix $A \in \Re^{n\times n}$ and the desired asymptotically stable matrix $\overline{A} \in \Re^{n\times n}$ have both the same Frobenius canonical form then the nonsingular matrix *M* is given by (4.4) and (4.5), respectively.

Example 4.1. For the unstable matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \tag{4.6}$$

and the desired asymptotically stable matrix

$$\overline{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \tag{4.7}$$

compute the matrix $M \in \Re^{2\times 2}$.

Using (4.4), (4.5), (4.6) and (4.7) we obtain

$$M = \overline{A}A^{-1} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -17 & -6 \end{bmatrix} \tag{4.8}$$

and

$$M = A^{-1}\overline{A} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} -6 & -3 \\ 0 & 1 \end{bmatrix}. \tag{4.9}$$

Now let us consider the following more general case. The matrix $A \in \Re^{n\times n}$ is nonsingular and unstable and the desired matrix $\overline{A} \in \Re^{n\times n}$ is asymptotically stable in the Frobenius canonical form. The desired matrix $M \in \Re^{n\times n}$ can be computed using (4.4) and (4.5).

Example 4.2. The matrices *A* and $\overline{A}$ have the forms

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 2 & 2 \end{bmatrix}, \quad \overline{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -27 & -27 & -5 \end{bmatrix}. \tag{4.10}$$

Note that the matrix *A* is unstable

$$\det[I_3 s - A] = \begin{vmatrix} s-2 & 0 & -1 \\ 0 & s+1 & 0 \\ 0 & -2 & s-2 \end{vmatrix} = (s+1)(s-2)^2 \tag{4.11}$$

and the matrix $\overline{A}$ is asymptotically stable

$$\det[I_3 s - \overline{A}] = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 27 & 27 & s+9 \end{vmatrix} = (s+3)^2. \tag{4.12}$$

Using (4.4) and (4.10) we obtain

$$M = \overline{A}A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -27 & -27 & -5 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -6.25 & 5.75 & -2.75 \\ 0 & 0 & -1 \\ 13.5 & -13.5 & 5.5 \end{bmatrix} \tag{4.13}$$

and using (4.5) and (4.10)

$$M = A^{-1}\overline{A} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -27 & -27 & -5 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0.5 \\ -13.5 & 22.5 & -22.5 \end{bmatrix} \tag{4.14}$$

Note that the matrices (4.13) and (4.14) are different and unstable.

From the above considerations we have the following conclusion:

Conclusion 4.1. For any nonsingular matrix $A \in \Re^{n \times n}$ of the linear continuous-time system (2.1) there exists the nonsingular matrix $M \in \Re^{n \times n}$ such that the matrix $\overline{A} \in \Re^{n \times n}$ has the desired eigenvalues.

5. STABILIZATION OF THE DISCRETE-TIME LINEAR SYSTEMS

Let $\Re_+^n$ be the set of n-dimensional vector with nonnegative components and $\Re_+^{n \times n}$ be the set of $n \times n$ matrices $A = [a_{ij}]$ with nonnegative entries $a_{ij} > 0$ for i,j = 1,...,n.

Definition 5.1.The linear system (2.11) is called positive if for any initial conditions $x_0 \in \Re_+^n$ the state vector $x_i \in \Re_+^n$ for *i* = 1,2,...

Theorem 5.1. [7] The linear system (2.11) is positive if and only if $A \in \Re_+^{n \times n}$.

Definition 5.2. The linear system (2.11) is called asymptotically stable if for any initial conditions $x_0 \in \Re_+^n$

$$\lim_{i \to \infty} x_i = 0 \tag{5.1}$$

Theorem 5.2. [6] The positive linear system (2.11) is asymptotically stable if and only if there exists a vector $\lambda \in \Re_+^n$ with all positive components $\lambda_i > 0$, *i* =1,...,*n* such that

$$A\lambda < \lambda \text{ and } A\lambda < \lambda . \tag{5.2}$$

Note that if the matrix $A = [a_{ij}] \in \Re^{n \times n}$ *i, j* =1,...,*n* then the matrix $A' = [|a_{ij}|] \in \Re_+^{n \times n}$.

Therefore, by Theorem 5.2, we have the following

Theorem 5.3. The matrix $A = [a_{ij}] \in \Re^{n \times n}$ is asymptotically stable if

$$\sum_{j=1}^{n} |a_{ij}| < 1 \text{ for } i = 1,\dots,n \tag{5.3a}$$

or

$$\sum_{i=1}^{n} |a_{ij}| < 1 \text{ for } j = 1,\dots,n. \tag{5.3b}$$

From Theorem 5.3 it follows that the matrix $M \in \Re^{n \times n}$ can be chosen so that the matrices *MA* and *AM* are asymptotically stable. Therefore, we have the following theorem.

Theorem 5.4. For unstable matrix $A \in \Re^{n \times n}$ of the discrete-time linear system (2.11) there exists a nonsingular matrix $M \in \Re^{n \times n}$ such that the matrices *MA* and *AM* are asymptotically stable.

Remark 5.1. If matrix *A* has some negative entries then we obtain more restrictive stability conditions comparing with the case with positive entries.

Example 5.1. Consider the discrete-time system (2.11) with the matrix

$$A = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \tag{5.4}$$

The matrix (5.4) is unstable since

$$\det[I_2 z - A] = \begin{vmatrix} z+2 & -1 \\ -1 & z-2 \end{vmatrix} = z^2 - 5 \tag{5.5}$$

and the eigenvalues of the matrix (5.4) are $z_1 = 5, \quad z_2 = -5$.

To stabilize the system we choose the matrix *M* in the form

$$M = \begin{bmatrix} 0.1 & 0.05 \\ 0.2 & 0.05 \end{bmatrix}. \tag{5.6}$$

In this case

$$MA = \begin{bmatrix} 0.1 & 0.05 \\ 0.2 & 0.05 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -0.15 & 1.2 \\ -0.35 & 0.3 \end{bmatrix} \tag{5.7}$$

and

$$AM = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0.1 & 0.05 \\ 0.2 & 0.05 \end{bmatrix} = \begin{bmatrix} 0 & -0.05 \\ 0.5 & 0.15 \end{bmatrix}. \tag{5.8}$$

The matrices (5.7) and (5.8) are asymptotically stable.

If in the matrix (5.4) the negative entry – 2 is substituted by its positive value 2, then we obtain

$$MA = \begin{bmatrix} 0.1 & 0.05 \\ 0.2 & 0.05 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0.25 & 0.2 \\ 0.45 & 0.3 \end{bmatrix} \tag{5.9}$$

and

$$AM = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0.1 & 0.05 \\ 0.2 & 0.05 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.15 \\ 0.5 & 0.15 \end{bmatrix}. \tag{5.10}$$

Note that the matrices (5.9) and (5.10) are also asymptotically stable.

A real matrix $A \in \Re^{n\times n}$ is called nilpotent if there exists a natural number (called the nilpotency index) $v \le n$ such that $A^{v-1} \ne 0$ and $A^{v} = 0$. The nilpotent matrix has only zero eigenvalues.

Theorem 5.5. If the matrix *A* in equation (2.11) is nilpotent with index *v* then its solution satisfies the condition

$$x_i = A^i x_0 = 0 \text{ for } i \ge v, v+1,... \tag{5.11}$$

Proof follows immediately from the definition of the nilpotent index *v* and that $A^i x_0 = 0$ for $i \ge v, v+1,...$ for any finite $x_0 \in \Re^n$.

Example 5.2. Compute the solution of the equation (2.11) for the following two nilpotent matrices

Case 1.

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}. \tag{5.12a}$$

Case 2.

$$A_2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.12b}$$

Case 1.The nilpotency index of the matrix (5.12a) is *v* = 2, since

$$A_1^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.13}$$

In this case for any finite initial condition $x_0 = 0$ we have

$$x_2 = A_1^2 x_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ for } i = 2,3,... \tag{5.14}$$

Case 2.The nilpotency index of the matrix (5.12b) is $v = 3$, since

$$A_2^3 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.15}$$

In this case for any finite initial conditions $x_0 = 0$ we have

$$x_3 = A_2^3 x_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ for } i = 2,3,... \tag{5.16}$$

Theorem 5.6. If the matrix $A \in \Re^{n\times n}$ is nonsingular then there exists a matrix $M \in \Re^{n\times n}$ such that

$$MA = A_{1n} \tag{5.17a}$$

and

$$AM = A_{2n} \tag{5.17b}$$

where $A_{1n} \in \Re^{n\times n}$ and $A_{2n} \in \Re^{n\times n}$ are given nilpotent matrices.

Proof. By assumption the matrix *A* is nonsingular and there exists its inverse A^{-1}. From (5.17a) we have

$$M = A_{1n}A^{-1} \tag{5.18}$$

and from (5.17b)

$$M = A^{-1}A_{2n} \tag{5.19}$$

respectively. This completes the proof.

Example 5.3. For unstable system with the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \tag{5.20}$$

find the matrices M_1 and M_2 for the nilpotent matrix

$$A_{1n} = A_{2n} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.21}$$

Using (5.18), (5.19), (5.20) and (5.21) we obtain

$$M = A_{1n}A^{-1} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -2 & -4 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}, \tag{5.22}$$

$$M = A^{-1}A_{2n} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix}. \tag{5.23}$$

Note that the matrices (5.22) and (5.23) are different.

6. CONCLUDING REMARKS

New approaches to the stabilization of the continuous-time and discrete-time linear systems have been proposed. The matrix algebraic equations with the Frobenius canonical matrices have been investigated (Theorem 4.1). It has been shown that for unstable matrix A of the discrete-time linear system (2.11) there exists a nonsingular matrix M such that:

1) the matrices MA and AM are both asymptotically stable (Theorem 5.4),
2) if the matrix A is nonsingular then there exists a matrix M such that the matrices MA and AM are nilpotent (Theorem 5.6).

The considerations have been illustrated by simple numerical examples. The considerations can be extended to the positive continuous-time and discrete-time linear systems, the fractional orders linear systems and to the 2-D linear systems described by the Roesser model.

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