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SHUFFLE ALGORITHM FOR FRACTIONAL DESCRIPTOR ROESSER TYPE CONTINUOUS-TIME LINEAR SYSTEMS

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Abstract – The shuffle algorithm is applied to analysis of the fractional descriptor Roesser type continuous-time linear systems. Using the shuffle algorithm the fractional descriptor linear system is reduced to the equivalent standard system and the system is decomposed into dynamic and static parts. Procedure for computation of the matrices of equivalent standard system and of the dynamical and static parts of the system is proposed.

Key words – fractional, descriptor, Roesser type, decomposition, dynamic and static parts

INTRODUCTION

Mathematical fundamentals of the fractional calculus are given in the monographs [27, 18, 22]. The properties of descriptor (singular) linear systems have been analysed in [4-8, 10, 11, 13, 15, 23, 27, 28]. The positive linear systems have been introduced and analysed in [9, 14, 18, 22]. The positive fractional linear systems have been investigated in [3, 17-20, 28]. Stability and stabilization of fractional linear systems has been investigated in the papers [3, 18, 22, 30]. The notion of practical stability of positive fractional discrete-time linear systems has been introduced in [18]. General response formula for CFD –fractional 2D continues linear systems described by the Roesser model is given in [28]. Some recent interesting results in fractional systems theory and its applications can be found in [18, 21, 28, 29]. The descriptor linear systems have been analysed by the use of the shuffle algorithm have been analysed in [24-26, 15]. The shuffle algorithm has be applied to compute the solution to the fractional descriptor linear system and to decompose the system into dynamic and static parts in [11]. In this paper the shuffle algorithm will be applied to the fractional descriptor Roesser type continuous-time linear system and procedures will be given for reduction the descriptor system to equivalent standard system and decomposed the system into dynamical and static parts.

The paper is organized as follows. In section 2 the elementary row and column operations and inverses of non-square matrices are recalled. The shuffle algorithm to the fractional descriptor linear Roesser type linear systems is applied in section 3 to reduce the descriptor system to equivalent standard system and to decompose the descriptor system into dynamic and static parts in section 4. Concluding remarks are given in section 5.

The following notations will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n\times n$ identity matrix.[1]

I. PRELIMINARIES

In the shuffle algorithm the following elementary operations on real matrices will be used [14]:

- Multiplication of any *i*-th row (column) by the number a. This operation will be denoted by $L[i\times a]$ for row operation and by $R[i\times a]$ for column operation.
- Addition to any *i*-th row (column) of the *j*-th row (column) multiplied by any number b. This operation will be denoted by $L[i+j\times b]$ for row operation and by $R[i+j\times b]$ for column operation.
- The interchange of rows will be denoted by $L[i, j]$, the interchange of columns by $R[i, j]$.

Lemma 1. Let the matrix $A\in\Re^{n\times m}$ has full row rank, $\mathrm{rank}A = n$. The set of its right inverse matrices is given by:

1) $$A_r = A^T[AA^T]^{-1} + (I_m - A^T[AA^T]^{-1}A)K_1 \quad (1a)$$

2) $$A_r = K_2[AK_2]^{-1} \quad (1b)$$

where T denotes the transpose and $K_1, K_2 \in \Re^{n\times m}$ ($\det AK_2 \neq 0$) are any real matrices.

Proof. By definition of the right inverse of the matrix we

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have

$$AA_r = A\left\{A^T[AA^T]^{-1} + (I_n - A^T[AA^T]^{-1}A)K_1\right\} = I_m + (A - A)K_1 = I_m. \quad (2)$$

Taking into account that K_2 is chosen so that $\det AK_2 \neq 0$ we obtain

$$AA_r = AK_2[AK_2]^{-1} = I_m . \square \quad (3)$$

II. FORMAT SHUFFLE ALGORITHM FOR FRACTIONAL DESCRIPTOR ROESSER TYPE LINEAR SYSTEMS

Consider the fractional descriptor Roesser type continuous-time linear system

$$E\begin{bmatrix} \dfrac{\partial^{\alpha_h} x^h(t_1,t_2)}{\partial t^{\alpha_h}} \\ \dfrac{\partial^{\alpha_v} x^v(t_1,t_2)}{\partial t^{\alpha_v}} \end{bmatrix} = A\begin{bmatrix} x^h(t_h,t_v) \\ x^v(t_h,t_v) \end{bmatrix} + Bu(t_h,t_v), \quad (4a)$$

$$E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

$$E_{11}, A_{11} \in \Re^{n_1 \times n_1}, \quad E_{22}, A_{22} \in \Re^{n_2 \times n_2},$$

$$B_1 \in \Re^{n_1 \times m}, \quad B_2 \in \Re^{n_2 \times m}$$

where $x^h(t_h,t_v) \in \Re^{n_1}$, $x^v(t_h,t_v) \in \Re^{n_2}$ are the horizontal and vertical state vectors, $u(t_h,t_v) \in \Re^m$ is the input vector and

$$\frac{d^\alpha x}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^\alpha} d\tau, \quad \dot{x}(\tau) = \frac{dx(\tau)}{d\tau}, \quad (4b)$$

$0 < \alpha < 1$, is the Caputo derivative of the order α,

$$\Gamma(x) = \int_0^\infty e^{-t} t^{\alpha-1} dt \quad (4c)$$

is the gamma function.

It is assumed that $\operatorname{rank} E < n$ and

$$\det\begin{bmatrix} E_{11}s_h^{\alpha_h} - A_{11} & E_{12}s_v^{\alpha_v} - A_{12} \\ E_{21}s_h^{\alpha_h} - A_{21} & E_{22}s_v^{\alpha_v} - A_{22} \end{bmatrix} = d_{n_1,n_2}(s_h^{\alpha_h})^{n_1}(s_v^{\alpha_v})^{n_2} + \dots + d_{11}s_h^{\alpha_h}s_v^{\alpha_v} + d_{10}s_h^{\alpha_h} + d_{01}s_v^{\alpha_v} + d_{00} \neq 0 \quad (5)$$

for some $s_h, s_v \in \mathbf{C}$ (the field of complex numbers).

In this case the equation (4a) has unique solution [22, 24, 25].

Performing elementary row operations on the array

$$[E \quad A \quad B] = \begin{bmatrix} E_{11} & E_{12} & A_{11} & A_{12} & B_1 \\ E_{21} & E_{22} & A_{21} & A_{22} & B_2 \end{bmatrix} \quad (6)$$

eliminating the linear depending rows of the matrices $[E_{11} \quad E_{12}]$ and $[E_{21} \quad E_{22}]$ we obtain

$$\begin{bmatrix} E_{11}^{(1)} & E_{12}^{(1)} & A_{11}^{(1)} & A_{12}^{(1)} & B_1^{(1)} \\ 0 & 0 & \overline{A}_{11}^{(1)} & 0 & \overline{B}_1^{(1)} \\ E_{21}^{(1)} & E_{22}^{(1)} & A_{21}^{(1)} & A_{22}^{(1)} & B_2^{(1)} \\ 0 & 0 & 0 & \overline{A}_{22}^{(1)} & \overline{B}_2^{(1)} \end{bmatrix} \quad (7)$$

and the equations

$$[E_{11}^{(1)} \quad E_{12}^{(1)}]\begin{bmatrix} \dfrac{\partial^{\alpha_h} x^h(t_h,t_v)}{\partial t^{\alpha_h}} \\ \dfrac{\partial^{\alpha_v} x^v(t_h,t_v)}{\partial t^{\alpha_v}} \end{bmatrix} = [A_{11}^{(1)} \quad A_{12}^{(1)}]\begin{bmatrix} x^h(t_h,t_v) \\ x^v(t_h,t_v) \end{bmatrix} + B_1^{(1)}u(t_h,t_v), \quad (8a)$$

$$0 = [\overline{A}_{11}^{(1)} \quad 0]\begin{bmatrix} x^h(t_h,t_v) \\ x^v(t_h,t_v) \end{bmatrix} + \overline{B}_1^{(1)}u(t_h,t_v), \quad (8b)$$

$$[E_{21}^{(1)} \quad E_{22}^{(1)}]\begin{bmatrix} \dfrac{\partial^{\alpha_h} x^h(t_h,t_v)}{\partial t^{\alpha_h}} \\ \dfrac{\partial^{\alpha_v} x^v(t_h,t_v)}{\partial t^{\alpha_v}} \end{bmatrix} = [A_{21}^{(1)} \quad A_{22}^{(1)}]\begin{bmatrix} x^h(t_h,t_v) \\ x^v(t_h,t_v) \end{bmatrix} + B_2^{(1)}u(t_h,t_v), \quad (8c)$$

$$0 = [0 \quad \overline{A}_{22}^{(1)}]\begin{bmatrix} x^h(t_h,t_v) \\ x^v(t_h,t_v) \end{bmatrix} + \overline{B}_2^{(1)}u(t_h,t_v). \quad (8d)$$

Differentiating the equation (8b) by $\dfrac{\partial^{\alpha_h}}{\partial t^{\alpha_h}}$ and equation (8d) by $\dfrac{\partial^{\alpha_v}}{\partial t^{\alpha_v}}$, we obtain

$$[E_{11}^{(1)} - \overline{A}_{11}^{(1)} \quad E_{12}^{(1)}]\begin{bmatrix} \dfrac{\partial^{\alpha_h} x^h(t_h,t_v)}{\partial t^{\alpha_h}} \\ \dfrac{\partial^{\alpha_v} x^v(t_h,t_v)}{\partial t^{\alpha_v}} \end{bmatrix} = [A_{11}^{(1)} \quad A_{12}^{(1)}]\begin{bmatrix} x^h(t_h,t_v) \\ x^v(t_h,t_v) \end{bmatrix} + B_1^{(1)}u(t_h,t_v) + \overline{B}_1^{(1)}\frac{\partial^{\alpha_h} u(t_h,t_v)}{\partial t^{\alpha_h}}, \quad (9a)$$

$$[E_{21}^{(1)} \quad E_{22}^{(1)} - \overline{A}_{22}^{(1)}]\begin{bmatrix} \dfrac{\partial^{\alpha_h} x^h(t_h,t_v)}{\partial t^{\alpha_h}} \\ \dfrac{\partial^{\alpha_v} x^v(t_h,t_v)}{\partial t^{\alpha_v}} \end{bmatrix}$$

$$= [A_{21}^{(1)} \quad A_{22}^{(1)}]\begin{bmatrix} x^h(t_h,t_v) \\ x^v(t_h,t_v) \end{bmatrix} \tag{9b}$$

$$+ B_2^{(1)} u(t_h,t_v) + \overline{B}_2^{(1)} \frac{\partial^{\alpha_v} u(t_h,t_v)}{\partial t^{\alpha_v}}.$$

The equations (9a) and (9b) can be written in the forms

$$E^{(1)}\begin{bmatrix} \dfrac{\partial^{\alpha_h} x^h(t_h,t_v)}{\partial t^{\alpha_h}} \\ \dfrac{\partial^{\alpha_v} x^v(t_h,t_v)}{\partial t^{\alpha_v}} \end{bmatrix} = A^{(1)}\begin{bmatrix} x^h(t_h,t_v) \\ x^v(t_h,t_v) \end{bmatrix}$$

$$+ B^{(1)} u(t_h,t_v) + \overline{B}^{(1)}\begin{bmatrix} \dfrac{\partial^{\alpha_h} u(t_h,t_v)}{\partial t^{\alpha_h}} \\ \dfrac{\partial^{\alpha_v} u(t_h,t_v)}{\partial t^{\alpha_v}} \end{bmatrix}, \tag{10a}$$

where

$$E^{(1)} = \begin{bmatrix} E_{11}^{(1)} - \overline{A}_{11}^{(1)} & E_{12}^{(1)} \\ E_{21}^{(1)} & E_{22}^{(1)} - \overline{A}_{22}^{(1)} \end{bmatrix},$$

$$A^{(1)} = \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} \\ A_{21}^{(1)} & A_{22}^{(1)} \end{bmatrix}, \tag{10b}$$

$$B^{(1)} = \begin{bmatrix} B_1^{(1)} \\ B_2^{(1)} \end{bmatrix}, \quad \overline{B}^{(1)} = \begin{bmatrix} \overline{B}_1^{(1)} \\ \overline{B}_2^{(1)} \end{bmatrix}.$$

Note that by the assumption (5) the matrices $E^{(1)}$, $\overline{A}^{(1)}$ have full row ranks and they have right inverses.
Finally we obtain the standard system

$$\begin{bmatrix} \dfrac{\partial^{\alpha_h} x^h(t_h,t_v)}{\partial t^{\alpha_h}} \\ \dfrac{\partial^{\alpha_v} x^v(t_h,t_v)}{\partial t^{\alpha_v}} \end{bmatrix} = \hat{A}^{(1)}\begin{bmatrix} x^h(t_h,t_v) \\ x^v(t_h,t_v) \end{bmatrix}$$

$$+ \hat{B}^{(1)} u(t_h,t_v) + \widetilde{B}^{(1)}\begin{bmatrix} \dfrac{\partial^{\alpha_h} u(t_h,t_v)}{\partial t^{\alpha_h}} \\ \dfrac{\partial^{\alpha_v} u(t_h,t_v)}{\partial t^{\alpha_v}} \end{bmatrix}, \tag{11}$$

where $\hat{A}^{(1)} = E_r^{(1)} A^{(1)}$, $\hat{B}^{(1)} = E_r^{(1)} B^{(1)}$, $\widetilde{B}^{(1)} = E_r^{(1)} \overline{B}^{(1)}$ and $E_r^{(1)}$, $\overline{A}_r^{(1)}$ are the right inverses defined by (1) of the matrices.
Therefore, the following theorem has been proved.

Theorem 1. The fractional descriptor Roesser type continuous linear system (4) satisfying the condition (5) can be reduced by the use of the shuffle algorithm to the standard system (11).

From the considerations we have following procedure for finding standard system (11) for the given descriptor system (4).

Procedure 1.

Step 1. Performing elementary row operations on (6) find (7).

Step 2. Using (1) compute the right inverses $E_r^{(1)}$, $\overline{A}_r^{(1)}$ of the matrices $E^{(1)}$, $\overline{A}^{(1)}$ and the matrices $\hat{A}^{(1)}$, $\hat{B}^{(1)}$, $\hat{B}_1^{(1)}$.

Step 3. Using (11) find the desired standard model of the system.

Example 1. For the given descriptor system (4) with the matrices

$$E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} = \left[\begin{array}{cc|ccc} 0 & 1 & 1 & 0 & 2 \\ 0 & -2 & -2 & 0 & -4 \\ \hline 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ 0 & 2 & 2 & 0 & 0 \end{array}\right],$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{cc|ccc} 0 & 1 & 1 & 1 & 0 \\ 0 & 2 & -2 & -2 & 0 \\ \hline -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{array}\right], \tag{12}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{array}\right], \quad n_1 = 2, \quad n_2 = 3,$$

using Procedure 1 find the corresponding standard system (10). It is easy to check that the descriptor system with (11) satisfies that assumption (5). Using Procedure 1 we obtain the following:

Step 1. Performing on the array

$$\begin{bmatrix} E_{11} & E_{12} & A_{11} & A_{12} & B_1 \\ E_{21} & E_{22} & A_{21} & A_{22} & B_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & -2 & 0 & -4 & 0 & 2 & -2 & -2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \tag{13}$$

the elementary operations: L[2+1x2], L[4+3] and L[4,5] we obtain

$$
\begin{bmatrix} E_{11}^{(1)} & E_{12}^{(1)} & A_{11}^{(1)} & A_{12}^{(1)} & B_1^{(1)} \\ 0 & 0 & \overline{A}_{11}^{(1)} & 0 & \overline{B}_1^{(1)} \\ E_{21}^{(1)} & E_{22}^{(1)} & A_{21}^{(1)} & A_{22}^{(1)} & B_2^{(1)} \\ 0 & 0 & 0 & \overline{A}_{22}^{(1)} & \overline{B}_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}. \tag{14}
$$

In this case the equations (8) have the form

$$
[0 \quad 1 \quad 1 \quad 0 \quad 2]\begin{bmatrix} \dfrac{\partial^{\alpha_h} x^h(t_h,t_v)}{\partial t^{\alpha_h}} \\ \dfrac{\partial^{\alpha_v} x^v(t_h,t_v)}{\partial t^{\alpha_v}} \end{bmatrix} = [0 \quad 1 \quad 1 \quad 1 \quad 0]\begin{bmatrix} x^h(t_h,t_v) \\ x^v(t_h,t_v) \end{bmatrix} + [1 \quad 0]u(t_h,t_v), \tag{15a}
$$

$$
0 = [0 \quad 4 \quad 0 \quad 0 \quad 0]\begin{bmatrix} x^h(t_h,t_v) \\ x^v(t_h,t_v) \end{bmatrix} + [2 \quad 1]u(t_h,t_v), \tag{15b}
$$

$$
\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dfrac{\partial^{\alpha_h} x^h(t_h,t_v)}{\partial t^{\alpha_h}} \\ \dfrac{\partial^{\alpha_v} x^v(t_h,t_v)}{\partial t^{\alpha_v}} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x^h(t_h,t_v) \\ x^v(t_h,t_v) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}u(t_h,t_v), \tag{15c}
$$

$$
0 = [0 \quad 0 \quad 1 \quad 1 \quad 0]\begin{bmatrix} x^h(t_h,t_v) \\ x^v(t_h,t_v) \end{bmatrix} + [0 \quad 1]u(t_h,t_v)\,. \tag{15d}
$$

Derivation (15b) and (15d) we obtain equations (9) in the form

$$
[0 \quad -3 \quad 1 \quad 0 \quad 2]\begin{bmatrix} \dfrac{\partial^{\alpha_h} x^h(t_h,t_v)}{\partial t^{\alpha_h}} \\ \dfrac{\partial^{\alpha_v} x^v(t_h,t_v)}{\partial t^{\alpha_v}} \end{bmatrix} = [0 \quad 1 \quad 1 \quad 1 \quad 0]\begin{bmatrix} x^h(t_h,t_v) \\ x^v(t_h,t_v) \end{bmatrix} + [1 \quad 0]u(t_h,t_v) + [2 \quad 1]\frac{\partial^{\alpha_h} u(t_h,t_v)}{\partial t^{\alpha_h}}, \tag{16a}
$$

$$
\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} \dfrac{\partial^{\alpha_h} x^h(t_h,t_v)}{\partial t^{\alpha_h}} \\ \dfrac{\partial^{\alpha_v} x^v(t_h,t_v)}{\partial t^{\alpha_v}} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x^h(t_h,t_v) \\ x^v(t_h,t_v) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}u(t_h,t_v) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\frac{\partial^{\alpha_v} u(t_h,t_v)}{\partial t^{\alpha_v}}. \tag{16b}
$$

Step 2. Combining (16) to the form (10a) we obtain matrices (10b) in the form

$$
E^{(1)} = \begin{bmatrix} 0 & -3 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 \end{bmatrix}, \quad A^{(1)} = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix},
$$

$$
B^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \overline{B}^{(1)} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{17}
$$

Choosing

$$
K_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{18}
$$

and using (1b) with (17) we compute

$$
E_r^{(1)} = \begin{bmatrix} 0 & 0.5 & 0 \\ -0.111 & 0.167 & 0.333 \\ 0.222 & 0.167 & 0.333 \\ 0 & 0.5 & 0 \\ 0.222 & 0.167 & 0.333 \end{bmatrix}, \quad \overline{A}_r^{(1)} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{19a}
$$

and

$$\hat{A}^{(1)}=E_r^{(1)}A^{(1)}=\begin{bmatrix} -0.5 & 0.5 & 0 & 0.5 & 0 \\ -0.167 & 0.389 & -0.111 & 0.056 & 0.333 \\ -0.167 & 0.722 & 0.222 & 0.289 & 0.333 \\ -0.5 & 0.5 & 0 & 0.5 & 0 \\ -0.167 & 0.722 & 0.222 & 0.389 & 0.333 \end{bmatrix},$$

$$\hat{B}^{(1)}=E_r^{(1)}B^{(1)}=\begin{bmatrix} 0 & 0 \\ 0.222 & 0 \\ 0.556 & 0 \\ 0 & 0 \\ 0.556 & 0 \end{bmatrix},$$

$$\widetilde{B}^{(1)}=E_r^{(1)}\overline{B}^{(1)}=\begin{bmatrix} 0 & 0 \\ -0.222 & 0.222 \\ 0.444 & 0.556 \\ 0 & 0 \\ 0.444 & 0.556 \end{bmatrix}. \tag{19b}$$

Step 3. Using (10) we have

$$\begin{bmatrix} \dfrac{\partial^{\alpha_1}x^h(t_h,t_v)}{\partial t^{\alpha_1}} \\ \dfrac{\partial^{\alpha_2}x^v(t_h,t_v)}{\partial t^{\alpha_2}} \end{bmatrix} =\begin{bmatrix} -0.5 & 0.5 & 0 & 0.5 & 0 \\ -0.167 & 0.389 & -0.111 & 0.056 & 0.333 \\ -0.167 & 0.722 & 0.222 & 0.289 & 0.333 \\ -0.5 & 0.5 & 0 & 0.5 & 0 \\ -0.167 & 0.722 & 0.222 & 0.389 & 0.333 \end{bmatrix}\begin{bmatrix} x^h(t_h,t_v) \\ x^v(t_h,t_v) \end{bmatrix} +\begin{bmatrix} 0 & 0 \\ 0.222 & 0 \\ 0.556 & 0 \\ 0 & 0 \\ 0.556 & 0 \end{bmatrix}u(t_h,t_v)+\begin{bmatrix} 0 & 0 \\ -0.222 & 0.222 \\ 0.444 & 0.556 \\ 0 & 0 \\ 0.444 & 0.556 \end{bmatrix}\begin{bmatrix} \dfrac{\partial^{\alpha_1}u(t_h,t_v)}{\partial t^{\alpha_1}} \\ \dfrac{\partial^{\alpha_2}u(t_h,t_v)}{\partial t^{\alpha_2}} \end{bmatrix}. \tag{20}$$

III. Decomposition of the Descriptor Systems into Dynamical and Static Parts

In this section the shuffle algorithm will be applied to the decomposition of the descriptor fractional Roesser type model into dynamical and static parts.

Let

$$\det\begin{bmatrix} E_{11}s_h^{\alpha_h}-A_{11} & E_{12}s_v^{\alpha_v}-A_{12} \\ E_{21}s_h^{\alpha_h}-A_{21} & E_{22}s_v^{\alpha_v}-A_{22} \end{bmatrix} =a_{d_1,d_2}(s_h^{\alpha_h})^{d_1}(s_v^{\alpha_v})^{d_2}+\ldots+a_{11}s_h^{\alpha_h}s_v^{\alpha_v} +a_{10}s_h^{\alpha_h}+a_{01}s_v^{\alpha_v}+a_{00},\ a_{d_1,d_2}\neq 0. \tag{21}$$

Performing elementary row operations on (6) eliminating the linear dependent rows of the matrices $[E_{11}\ \ E_{12}]$, $[E_{21}\ \ E_{22}]$ we obtain (7). If

$$\operatorname{rank}\begin{bmatrix} E_{11}^{(1)} & E_{12}^{(1)} \end{bmatrix}<d_1 \text{ and } \operatorname{rank}\begin{bmatrix} E_{21}^{(1)} & E_{22}^{(1)} \end{bmatrix}<d_2 \tag{22}$$

then by shuffles of (7) we obtain

$$\begin{bmatrix} E_{11}^{(1)} & E_{12}^{(1)} & A_{11}^{(1)} & A_{12}^{(1)} & B_1^{(1)} & 0 \\ \overline{A}_{11}^{(1)} & 0 & 0 & 0 & 0 & \overline{B}_1^{(1)} \\ E_{21}^{(1)} & E_{22}^{(1)} & A_{21}^{(1)} & A_{22}^{(1)} & B_2^{(1)} & 0 \\ 0 & \overline{A}_{22}^{(1)} & 0 & 0 & 0 & \overline{B}_2^{(1)} \end{bmatrix}. \tag{23}$$

Performing suitable elementary row operations on (23) we obtain

$$\begin{bmatrix} E_{11}^{(2)} & E_{12}^{(2)} & A_{11}^{(2)} & A_{12}^{(2)} & \overline{B}_1^{(2)} & \overline{B}_1^{(2)} \\ 0 & 0 & \overline{A}_{11}^{(2)} & 0 & \overline{B}_2^{(2)} & \overline{B}_3^{(2)} \\ E_{21}^{(2)} & E_{22}^{(2)} & A_{21}^{(2)} & A_{22}^{(2)} & \overline{B}_3^{(2)} & \overline{B}_4^{(2)} \\ 0 & 0 & 0 & \overline{A}_{22}^{(2)} & \overline{B}_5^{(2)} & \overline{B}_6^{(2)} \end{bmatrix}. \tag{24}$$

If $\operatorname{rank}\begin{bmatrix} E_{11}^{(2)} & E_{12}^{(2)} \end{bmatrix}<d_1$ and $\operatorname{rank}\begin{bmatrix} E_{21}^{(2)} & E_{22}^{(2)} \end{bmatrix}<d_2$ then we repeat the shuffles. If the assumption (2) is satisfied then after q steps $(q<n_1,n_2)$ we obtain the matrices $\begin{bmatrix} E_{11}^{(q)} & E_{12}^{(q)} \end{bmatrix}$, $\begin{bmatrix} E_{21}^{(q)} & E_{22}^{(q)} \end{bmatrix}$ satisfying the condition

$$\operatorname{rank}\begin{bmatrix} E_{11}^{(q)} & E_{12}^{(q)} \end{bmatrix}=d_1 \text{ and } \operatorname{rank}\begin{bmatrix} E_{21}^{(q)} & E_{22}^{(q)} \end{bmatrix}=d_2. \tag{25}$$

Note that if the conditions (25) are satisfied, then the matrices $\begin{bmatrix} E_{11}^{(q)} & E_{12}^{(q)} \end{bmatrix}$, $\begin{bmatrix} E_{21}^{(q)} & E_{22}^{(q)} \end{bmatrix}$ have right inverses and using (1b) from (24) we obtain

$$E_r^{(q)}=\frac{1}{E_{11}^{(q)}E_{22}^{(q)}-E_{12}^{(q)}E_{21}^{(q)}}\begin{bmatrix} E_{11}^{(q)} & E_{12}^{(q)} \\ -E_{21}^{(q)} & E_{22}^{(q)} \end{bmatrix},\quad \overline{A}_r^{(q)}=\frac{1}{\overline{A}_{11}^{(q)}\overline{A}_{22}^{(q)}}\begin{bmatrix} \overline{A}_{11}^{(1)} & 0 \\ 0 & \overline{A}_{22}^{(1)} \end{bmatrix}. \tag{26}$$

Therefore the following theorem has been proved.

Theorem 2. The fractional descriptor Roesser type continuous-time linear systems (4) satisfying the condition (5) can be decomposed by the use of the shuffle algorithm into the dynamical part (9a) and static part (9b).

IV. Concluding Remarks

The shuffle algorithm has been applied to analysis of the fractional descriptor Roesser type continuous-time linear systems. Procedure for computation of the equivalent standard system for given descriptor one has been proposed (Procedure 1).The descriptor Roesser model has been decomposed into the dynamic and static parts (Theorem 2). The procedure has been demonstrated on numerical example. The considerations can be extended to fractional descriptor discrete-time linear systems and to different fractional orders descriptor continuous-time and discrete-time linear systems.

ALGORYTM PRZESUWANIA DLA LINIOWYCH UKŁADÓW DESKRYPTOROWYCH NIECAŁKOWITEGO RZĘDU CIĄGŁYCH TYPU ROESSERA

Algorytm przesuwania jest stosowany do analizy ułamkowych deskryptorowych układów liniowych ciągłych typu Roesser. Stosując algorytm przesuwania, liniowy układ deskryptorowy niecałkowitego rzędu jest redukowany do równoważnego układu standardowego, oraz jest rozkładany na część dynamiczną i statyczną. Zaproponowano procedurę obliczania macierzy równoważnego układu standardowego oraz metodę obliczania części dynamicznej i statycznej.

Słowa kluczowe: niecałkowity rząd, deskryptor, dekompozycja, część statyczna i dynamiczna

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